

A LOWER BOUND FOR GENERALIZED DOMINATING NUMBERS

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ABSTRACT. We show a new proof for the fact that when κ and λ are infinite cardinals satisfying $\lambda^\kappa = \lambda$, the cofinality of the set of all functions from λ to κ ordered by everywhere domination is 2^λ . An earlier proof was a consequence of a result about independent families of functions. The new proof follows directly from the main theorem we present: for every $A \subseteq \lambda$ there is a function $f : {}^\kappa\lambda \rightarrow \kappa$ such that whenever M is a transitive model of ZF such that ${}^\kappa\lambda \subseteq M$ and some $g : {}^\kappa\lambda \rightarrow \kappa$ in M dominates f , then $A \in M$. That is, “constructibility can be reduced to domination”.

1. INTRODUCTION

Given a partially ordered set $\mathbb{P} = \langle P, \leq_P \rangle$, let $\text{cf}(\mathbb{P})$ be the cofinality of \mathbb{P} . That is, $\text{cf}(\mathbb{P}) = \min\{|A| : A \subseteq P \text{ and } (\forall p \in P)(\exists a \in A) p \leq_P a\}$. Let κ and λ be infinite cardinals. Regarding κ as a partially ordered set $\langle \kappa, \leq \rangle$, it is natural to wonder about the structure of the product of this ordering with itself λ many times. This is the same as the partially ordered set $\langle {}^\lambda\kappa, \leq \rangle$ of all functions from λ to κ , ordered by

$$(\forall f, g \in {}^\lambda\kappa) [f \leq g \iff (\forall \alpha < \lambda) f(\alpha) \leq g(\alpha)].$$

This is referred to as the *everywhere domination* ordering. When $f \leq g$, for brevity we will just say g dominates f . We will often write ${}^\lambda\kappa$ instead of $\langle {}^\lambda\kappa, \leq \rangle$ when no confusion should arise.

Of course, we could be “more general” and consider the everywhere domination ordering of all functions from an arbitrary set X to κ , but that is isomorphic to $\langle {}^{|X|}\kappa, \leq \rangle$. Nevertheless, it will sometimes be more convenient notationally to have the domains be arbitrary sets rather than cardinals, so we will do this freely.

We may want to investigate the cofinality $\text{cf}({}^\lambda\kappa)$ of the partially ordered set ${}^\lambda\kappa$. Without loss of generality, we can take κ to be regular. When κ is regular and $> \lambda$, it is clear that $\text{cf}({}^\lambda\kappa) = \kappa$ (because the set

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of constant functions is cofinal). Thus, we might as well assume $\kappa \leq \lambda$. When this happens, of course $\text{cf}(\lambda\kappa) \leq |\lambda\kappa| = 2^\lambda$.

There is also the partial ordering $\langle \lambda\kappa, \leq^* \rangle$ of all functions from λ to κ by *eventual domination*, where $f \leq^* g$ iff

$$(\exists \alpha < \lambda)(\forall \beta \geq \alpha) f(\beta) \leq g(\beta).$$

In the literature, this is often investigated more than everywhere domination, so we will spend a few moments to explain how the two notions are connected. Of course,

$$\text{cf}\langle \lambda\kappa, \leq^* \rangle \leq \text{cf}\langle \lambda\kappa, \leq \rangle.$$

A straightforward diagonalization shows

$$\lambda^+ \leq \text{cf}\langle \lambda\kappa, \leq^* \rangle.$$

Notice that for a regular λ ,

$$\text{cf}\langle \lambda\lambda, \leq^* \rangle = \text{cf}\langle \lambda\lambda, \leq \rangle,$$

because if we take a set \mathcal{F} cofinal in $\langle \lambda\lambda, \leq^* \rangle$ and replace each $f \in \mathcal{F}$ with the set of functions of the form

$$\alpha \mapsto \max\{f(\alpha), \beta\}$$

for some $\beta < \lambda$, we get a set cofinal in $\langle \lambda\lambda, \leq \rangle$ of size $|\mathcal{F}|$. However, the same trick *cannot* be used to argue $\text{cf}\langle \lambda\kappa, \leq^* \rangle = \text{cf}\langle \lambda\kappa, \leq \rangle$ when λ is a cardinal and κ is some regular cardinal $< \lambda$. There are other relationships between everywhere domination and eventual domination, for example when λ is regular,

$$\text{cf}\langle \lambda\kappa, \leq \rangle = \text{cf}\langle \lambda\kappa, \leq^* \rangle \cdot \sum_{\alpha < \lambda} \text{cf}\langle \alpha\kappa, \leq \rangle.$$

Also, for any λ ,

$$\text{cf}\langle \lambda\kappa, \leq \rangle \leq \text{cf}\langle \lambda^+\kappa, \leq^* \rangle$$

(more generally, for any $\mu \geq \lambda^+$, $\text{cf}\langle \lambda\kappa, \leq \rangle \leq \text{cf}\langle \mu\kappa, \leq^* \rangle$). Putting the above two relationships together,

$$\text{cf}\langle \lambda^+\kappa, \leq \rangle = \text{cf}\langle \lambda^+\kappa, \leq^* \rangle.$$

For results about alternative notions of eventual domination, see [10].

The case where $\kappa = \lambda$ is investigated in [4] by Cummings and Shelah. They show, as part of a more general result, that for a regular λ satisfying $\lambda^{<\lambda} = \lambda$, there is a λ -closed and λ^+ -c.c. forcing which forces $\text{cf}(\lambda\lambda) < 2^\lambda$.

In [12], Szymański determines whether $\text{cf}(\lambda\kappa) = 2^\lambda$ in all cases where κ is regular, $\kappa \leq \lambda < 2^\omega$, and 2^ω is real-valued measurable. Specifically,

he shows that under these assumptions, $\text{cf}({}^\omega\omega) < 2^\omega$, $\text{cf}({}^\lambda\omega) = 2^\lambda$ when $\omega < \lambda$, and $\text{cf}({}^\lambda\kappa) < 2^\lambda$ when $\omega < \kappa \leq \lambda$.

In [8], Jech and Prikry essentially show that whenever I is a κ^+ -complete ideal on λ and if there is a family \mathcal{F} of pairwise I -disjoint functions from λ to κ , then $|\mathcal{F}| \leq \text{cf}({}^\lambda\kappa)$. They then show various situations in which one might have such a family. They also begin to investigate the case when 2^ω is real-valued measurable.

A slightly different approach than using a family of I -almost disjoint functions for some ideal I is to use a family of sufficiently independent functions. We illustrate this connection in the next section, and show the consequence that if $\lambda^\kappa = \lambda$, then $\text{cf}({}^\lambda\kappa) = 2^\lambda$. In the section after that, we present our main theorem which helps to clarify the nature of the domination relation of functions from λ to κ when $\lambda^\kappa = \lambda$.

2. INDEPENDENT FAMILIES OF FUNCTIONS

Definition 2.1. Let λ , κ , and ν be infinite cardinals. A family $\mathcal{F} \subseteq {}^\lambda\kappa$ is said to be ν -independent iff

$$(\forall F \in [\mathcal{F}]^{<\nu})(\forall \varphi : F \rightarrow \kappa)(\exists x \in \lambda)(\forall f \in F) f(x) = \varphi(f).$$

We will now recall an old result. For the sake of this paragraph, let $I(\lambda, \kappa, \nu, \mu)$ be the statement “there exists a ν -independent family $\mathcal{F} \subseteq {}^\lambda\kappa$ of size μ ”. $I(\omega, 2, \omega, 2^\omega)$ and $I(2^\omega, 2, \omega, 2^{2^\omega})$ were both shown in [6]. For arbitrary infinite λ , $I(\lambda, 2, \omega, 2^\lambda)$ was shown in [7]. For infinite cardinals λ and κ such that $2^{<\kappa} \leq \lambda$, $I(\lambda, 2, \kappa, 2^\lambda)$ was shown in [13]. Finally, for infinite cardinals λ and κ such that $\lambda^{<\kappa} = \lambda$, $I(\lambda, \lambda, \kappa, 2^\lambda)$ was shown in [5]. We state this last result as the theorem below. For a proof of this theorem, see (a) \Rightarrow (d) of Theorem 3.16 in [3]. See also the end of Chapter 3 in [3] for more information.

Theorem 2.2. *If $\lambda^\kappa = \lambda$, then there is a κ^+ -independent family of 2^λ functions from λ to κ . More generally, if $\lambda^{<\kappa} = \lambda$, then there is a κ -independent family of 2^λ functions from λ to κ .*

Immediately, this gives us the desired bound:

Corollary 2.3. *If $\lambda^\kappa = \lambda$, then $\text{cf}({}^\lambda\kappa) = 2^\lambda$.*

Proof. Assuming $\lambda^\kappa = \lambda$, let \mathcal{F} be a κ^+ -independent family of 2^λ functions from λ to κ . Since this family is κ^+ -independent, every size κ subset is unbounded. Suppose, towards a contradiction, that there is some size $< 2^\lambda$ family $\mathcal{D} \subseteq {}^\lambda\kappa$ that is cofinal in ${}^\lambda\kappa$. Note that $\kappa < 2^\lambda$. By the pigeon hole principle, there is some $g \in \mathcal{D}$ which dominates at least κ members of \mathcal{F} . This contradicts every size κ subset of \mathcal{F} being unbounded. \square

This result was probably known, but the author could find no reference for it. An equivalent and amusing way to write this result is as follows:

$$\text{cf}({}^{(\lambda^\kappa)}\kappa) = 2^{\lambda^\kappa}.$$

As a special case of this corollary, we have the following:

Corollary 2.4. *The cofinality of the set of all functions from \mathbb{R} to ω ordered by everywhere domination is 2^{2^ω} .*

The last corollary gives us a different proof of a well-known result (which is attributed to Kunen in [8]):

Corollary 2.5. *CH implies $\text{cf}({}^{\omega_1}\omega) = 2^{\omega_1}$.*

There is a more general consequence of the theorem above which can be stated using the language of challenge response relations (see section 4 of [1]):

Proposition 2.6. *Suppose $\mathcal{R} = \langle R_-, R_+, R \rangle$ is a challenge response relation. Suppose $\kappa = ||\mathcal{R}^\perp||$. Let λ be a cardinal such that $\lambda^\kappa = \lambda$. Let $\tilde{\mathcal{R}} := \langle {}^\lambda R_-, {}^\lambda R_+, \tilde{R} \rangle$ be the conjunction of \mathcal{R} with itself λ many times. That is, $f \tilde{R} g$ iff $(\forall x \in \lambda) f(x) R g(x)$. Then $||\tilde{\mathcal{R}}|| = 2^\lambda$. In fact, there is a set $\mathcal{F} \subseteq {}^\lambda R_-$ of size 2^λ such that for every size κ subset A' of \mathcal{F} , there is no $g \in {}^\lambda R_+$ such that $(\forall f \in A') f \tilde{R} g$.*

Proof. Let $A = \{a_\alpha : \alpha < \kappa\} \subseteq R_-$ be a set of size κ such that there is no single $b \in R_+$ such that $(\forall \alpha < \kappa) a_\alpha R b$. Using Theorem 2.2, we obtain a set $\mathcal{F} = \{f_\beta : \beta < 2^\lambda\} \subseteq {}^\lambda R_-$ of size 2^λ such that for every injection $i : \kappa \rightarrow 2^\lambda$, there exists an $x \in \lambda$ such that

$$(\forall \alpha < \kappa) f_{i(\alpha)}(x) = a_\alpha.$$

The set \mathcal{F} is as desired. \square

3. THE MAIN THEOREM

Let κ and λ be infinite cardinals. The theorem below can be remembered as “for every $A \subseteq \lambda$ there is a function f from ${}^\kappa \lambda$ to κ such that A is constructible from ${}^\kappa \lambda$ and any g which dominates f ”.

Main Theorem 3.1. *For every $A \subseteq \lambda$ there is a function $f : {}^\kappa \lambda \rightarrow \kappa$ such that whenever M is a transitive model of ZF such that ${}^\kappa \lambda \subseteq M$ and some $g : {}^\kappa \lambda \rightarrow \kappa$ in M dominates f , then $A \in M$.*

Proof. Fix $A \subseteq \lambda$. Define f by

$$f(x) := \begin{cases} 0 & \text{if } (\forall \alpha < \kappa) x(\alpha) \notin A, \\ \alpha + 1 & \text{if } x(\alpha) \in A \text{ but } (\forall \beta < \alpha) x(\beta) \notin A. \end{cases}$$

Let M be a transitive model of ZF such that ${}^\kappa\lambda \subseteq M$ and $A \notin M$. Suppose, towards a contradiction, that there is some $g \in M$ that dominates f . Let B be the set

$$B := \{t \in {}^{<\kappa}\lambda : g(x) \geq \text{Dom}(t) \text{ for all } x \text{ extending } t\}.$$

Notice that $B \in M$.

For all $a \in \lambda$, $a \in A$ implies $\langle a \rangle \in B$. Thus, there must be some $a_0 \in \lambda$ such that $a_0 \notin A$ but $\langle a_0 \rangle \in B$. If there was not, then A could be defined in M by $A = \{a \in \lambda : \langle a \rangle \in B\}$, which would contradict the fact that $A \notin M$.

Next, for all $a \in \lambda$, $a \in A$ implies $\langle a_0, a \rangle \in B$. Thus, by similar reasoning as before, there must be some $a_1 \in \lambda$ such that $a_1 \notin A$ but $\langle a_0, a_1 \rangle \in B$. Continuing like this, we can construct a sequence $x \in {}^\kappa\lambda$ such that $(\forall \alpha < \kappa) x \restriction \alpha \in B$. This means that $(\forall \alpha < \kappa) g(x) \geq \alpha$. This contradicts g being well-defined at x . \square

For demonstration purposes, we show how this also implies Corollary 2.3:

Corollary 3.2. *If $\lambda^\kappa = \lambda$, then $\text{cf}({}^\lambda\kappa) = 2^\lambda$.*

Proof. Since there is a bijection between λ and ${}^\kappa\lambda$, it suffices to show that the cofinality of the set of all functions from ${}^\kappa\lambda$ to κ is at least 2^λ . Consider an arbitrary family \mathcal{A} of functions from ${}^\kappa\lambda$ to κ of size $< 2^\lambda$. We will show that it is not dominating.

Let

$$Z := \mathcal{A} \cup {}^\kappa\lambda \cup \lambda.$$

Let $E \prec V^1$ be such that $Z \subseteq E$ and $|E| < 2^\lambda$. Such an E exists because

$$|Z| = \max\{|\mathcal{A}|, \lambda\} < 2^\lambda.$$

Let $M := \pi(E)$ be the transitive collapse of E . Since ${}^\kappa\lambda \subseteq E$, we have $\pi(g) = g$ for all $g \in \mathcal{A}$. Hence, $\mathcal{A} \subseteq M$. Since $|M| < 2^\lambda$, there is some $A \in \mathcal{P}(\lambda) - M$. We may now apply the main lemma to get that there is some f not dominated by any member of M . In particular, such an f is not dominated by any member of \mathcal{A} . This completes the proof. \square

Notice that the main theorem required ${}^\kappa\lambda \subseteq M$. That is, M contains the domains of the functions involved. Dropping the requirement not only weakens the conclusion in the obvious way, but it also forces us to consider the possibility that the set B has no length κ branch in M , which breaks the proof. We can remove the ${}^\kappa\lambda \subseteq M$ assumption if we make various special modifications, and we will present several of them.

¹Instead of using V , we could use $H(\theta)$ for some appropriate regular cardinal θ .

In the case that $\kappa = \omega$, we can use the fact that well-foundedness of trees is absolute. This next theorem can be remembered as “for every $A \subseteq \lambda$ there is a function f from ${}^\omega\lambda$ to ω such that A is constructible from λ and any g which dominates f ”.

Theorem 3.3. *For every $A \subseteq \lambda$ there is a function $f : {}^\omega\lambda \rightarrow \omega$ such that whenever M is a transitive model of ZF such that $\lambda \in M$ and some $g : ({}^\omega\lambda)^M \rightarrow \omega$ in M satisfies*

$$(\forall x \in ({}^\omega\lambda)^M) f(x) \leq g(x),$$

then $A \in M$.

Proof. Fix $A \subseteq \lambda$. Define f as is done in the main theorem. Let M be a transitive model of ZF such that $\lambda \in M$ and which contains some appropriate g . Assume, towards a contradiction, that $A \notin M$. Define the set $B \subseteq {}^{<\omega}\lambda$ as is done in the proof of Theorem 3.1, except that the x in the definition ranges over elements of $({}^\omega\lambda)^M$ extending t . We have $B \in M$. Let $T \subseteq {}^{<\omega}\lambda$ be those elements of B all of whose initial segments are also in B . Note that $T \in M$. We may repeat the proof of the main lemma to get that T has a path in V . Since well-foundedness is absolute, there is some $x' \in [T] \cap M$. This x' witnesses that $(\forall n \in \omega) g(x') \geq n$, which is a contradiction. \square

If $\kappa > \omega$, there does not seem to be an obvious way to extend the last lemma, even if κ has some large cardinal property. However, if we are willing to sacrifice the sharpness of “a function f such that all dominators of f can *construct* A ”, then we can use an elementary substructure argument to fill in the part of the proof where we need B to have a length κ path in M . We also need the substructure M to include ${}^{<\kappa}\lambda$ for technical reasons which seem unavoidable. We state the next theorem but do not prove it, as it is very similar to the ones above.

Theorem 3.4. *For every $A \subseteq \lambda$ there is a function $f : {}^\kappa\lambda \rightarrow \kappa$ such that whenever $\langle M, \in \rangle \prec V$ is such that ${}^{<\kappa}\lambda \subseteq M$ and some $g : {}^\kappa\lambda \rightarrow \kappa$ in M satisfies*

$$(\forall x \in {}^\kappa\lambda) f(x) \leq g(x),$$

then $A \in M$.

If we assume that κ is weakly compact in M , then we can build a function from ${}^\kappa 2$ to κ that can only be dominated by a function in M if $A \in M$:

Theorem 3.5. *For every $a \in {}^\kappa 2$ there is a function $f : {}^\kappa 2 \rightarrow \kappa$ such that whenever M is a transitive model of ZF such that $\kappa \in M$, ${}^{<\kappa} 2 \subseteq M$, $(\kappa \text{ is weakly compact})^M$, and some $g : ({}^\kappa 2)^M \rightarrow \kappa$ in M satisfies*

$$(\forall x \in ({}^\kappa 2)^M) f(x) \leq g(x),$$

then $a \in M$.

Proof. Fix κ and $a \in {}^\kappa 2$. Let f be the function

$$f(x) := \begin{cases} \alpha & \text{if } x(\alpha) \neq a(\alpha) \text{ but } (\forall \beta < \alpha) x(\beta) = a(\beta), \\ 0 & \text{if } x = a. \end{cases}$$

Let M and g be as in the statement of the theorem. Assume, towards a contradiction, that $a \notin M$. Define B similarly as before:

$$B := \{t \in {}^{<\kappa} 2 : g(x) \geq \text{Dom}(t) \text{ for all } x \in ({}^\kappa 2)^M \text{ extending } t\}.$$

We have $B \in M$. Just as in Theorem 3.3, define $T \subseteq {}^{<\kappa} 2$ to be the set of elements of B all of whose initial segments are also in B . We have $T \in M$. Note also that since $(\kappa \text{ is strongly inaccessible})^M$, we have $(T \text{ is a } \kappa\text{-tree})^M$.

Now, since $a \notin M$, one can see that $a \restriction \alpha$ is in M for each $\alpha < \kappa$. In particular, T has height κ . This is calculated in V , but it is clearly absolute, so $(T \text{ has height } \kappa)^M$. Since $(T \text{ is a } \kappa\text{-tree})^M$, there must be some $x' \in [T] \cap M$. As in Theorem 3.3, this x' witnesses that $(\forall \alpha < \kappa) g(x') \geq \alpha$, which is a contradiction. \square

4. A CONSEQUENCE

We can use the main theorem of the last section to obtain a result that is incomparable to Proposition 2.6. In the language of [1], this proposition is saying there exists a morphism between two relations. We get this result because the main theorem of last section is really saying that a morphism exists between a certain domination relation and the constructibility relation:

Proposition 4.1. *Let $\langle P, \leq_P \rangle$ be a partial ordering and let κ be the smallest size of an unbounded subset of P ($\kappa = \mathfrak{b}\langle P, \leq_P \rangle$). Let λ be a cardinal and assume $|P| \leq 2^\lambda$. Let $\mathcal{F} := \langle {}^\lambda P, \leq_{\lambda P} \rangle$ be the partial ordering of all functions from λ to P given by $f \leq_{\lambda P} g$ iff*

$$(\forall \alpha < \lambda) f(\alpha) \leq_P g(\alpha).$$

Assume also that $\lambda^\kappa = \lambda$. Then $\text{cf}(\mathcal{F}) = 2^\lambda$. Moreover, there is a function $\phi^- : {}^\lambda \kappa \rightarrow {}^\lambda P$ and there is a function $\phi^+ : {}^\lambda P \rightarrow {}^\lambda \kappa$ such that

$$(\forall g \in {}^\lambda \kappa)(\forall f \in {}^\lambda P) \phi^-(g) \leq_{\lambda P} f \Rightarrow g \leq \phi^+(f).$$

Proof. Since $|P| \leq 2^\lambda$, we have ${}^\lambda P \leq (2^\lambda)^\lambda = 2^\lambda$, so $\text{cf}(\mathcal{F}) \leq 2^\lambda$. By Corollary 2.3, since $\lambda^\kappa = \lambda$, we have $\text{cf}({}^\lambda \kappa) = 2^\lambda$. Once we define the appropriate functions ϕ^- and ϕ^+ , it will follow that $\text{cf}({}^\lambda \kappa) \leq \text{cf}(\mathcal{F})$, and so $\text{cf}(\mathcal{F}) = 2^\lambda$.

Using induction, we can construct an unbounded chain $\langle a_\alpha : \alpha < \kappa \rangle$ in $\langle P, \leq_P \rangle$. That is, $(\forall \alpha < \beta < \kappa) a_\alpha \leq_P a_\beta$ and there is no $a \in P$ such that $(\forall \alpha < \kappa) a_\alpha \leq_P a$. Let $\phi^+ : {}^\lambda P \rightarrow {}^\lambda \kappa$ be the function

$$\phi^+(f) := (x \mapsto \min\{\alpha < \kappa : a_\alpha \not\leq_P f(x)\}).$$

Let $\phi^- : {}^\lambda \kappa \rightarrow {}^\lambda P$ be the function

$$\phi^-(g) := (x \mapsto a_{g(x)}).$$

These functions are as desired. \square

As an example of how to use this proposition, let P be the set of Lebesgue measure zero Borel subsets of \mathbb{R} . Let $A \leq_P B$ iff $A \subseteq B$. A straightforward argument shows that the smallest size κ of an unbounded subset of P is $\text{add}(\mathcal{L})$ (the additivity of Lebesgue measure, which appears in Cichoń's Diagram). Let $\lambda = 2^\omega$. Since every element of P is Borel, $|P| = \lambda$, but we need only that $|P| \leq 2^\lambda$. If $\lambda^\kappa = \lambda$, then $\text{cf}(\mathcal{F}) = 2^\lambda$. That is, if $2^{\text{add}(\mathcal{L})} = 2^\omega$, then $\text{cf}(\mathcal{F}) = 2^{2^\omega}$.

5. INTERPRETATION OF MAIN THEOREM

Within this section, whenever B_1, \dots, B_n are sets, let $M(B_1, \dots, B_n)$ refer to the smallest transitive model of ZF which contains B_1, \dots, B_n as elements. This is well-defined, although it could be a proper class. To say that $A \in M(B_1, \dots, B_n)$ is to say that A is constructible from B_1, \dots, B_n in a certain sense.

The main theorem can now be seen as the following statement:

$$(\forall A \subseteq \lambda)(\exists f : {}^\kappa \lambda \rightarrow \kappa)(\forall g : {}^\kappa \lambda \rightarrow \kappa)[f \leq g \Rightarrow A \in M(g, {}^\kappa \lambda)].$$

In the language of [1], this is saying there is a morphism from the domination relation of functions from ${}^\kappa \lambda$ to κ to the relation R defined by $ARg : \Leftrightarrow A \in M(g, {}^\kappa \lambda)$.

This is analogous to the situation with functions from ω to ω . There, however, we have only that for every *hyperarithmetical* $A \subseteq \omega$, there is a function $f : \omega \rightarrow \omega$ such that if a function $g : \omega \rightarrow \omega$ everywhere dominates f , then A is Turing reducible to g . Of course, changing “everywhere dominates” to “eventually dominates” makes no difference, because making finite modifications to a function does not change its Turing degree. See [2], [11], and [9] for more details.

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